

Matrix Equivalence and Isomorphism of Modules

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ABSTRACT

It is well known that if A and B are $n \times m$ matrices over a ring R , then $\text{coker } A \cong \text{coker } B$ does not imply A and B are equivalent. An elementary proof is given that the implication does hold if 1 is in the stable range of R . Furthermore, for certain R (including commutative rings), if A is block diagonal and B is block upper triangular with the same diagonal blocks as A , then $\text{coker } A \cong \text{coker } B$ implies A and B are equivalent under a special equivalence. This extends results of Roth and Gustafson. As a corollary, a theorem on decomposition of modules is obtained.

1. INTRODUCTION

Let R be a ring with 1, and denote by $R_{n \times m}$ the set of $n \times m$ matrices over R . Let $R_n = R_{n \times n}$, and $\text{Gl}_n(R)$ be the set of invertible matrices in R_n . A and $B \in R_{n \times m}$ are called equivalent if $UA = BV$ for some $U \in \text{Gl}_n(R)$ and $V \in \text{Gl}_m(R)$. A matrix $A \in R_{n \times m}$ determines a (right) R -module, $\text{coker } A = R^n / AR^m$, where A is considered as a map (on the space of column vectors) from R^m to R^n . If A and B are equivalent, then clearly $\text{coker } A \cong \text{coker } B$. The converse is not true in general. Levy and Robson [13, Theorem 4.3] proved that this does hold for semiperfect rings (see Section 2). In Section 2, this result is extended to rings with 1 in the stable range. These include algebras S over a local ring and are in general not semiperfect. Some applications are then given. For example, a result of Goodearl and Warfield [7] is obtained, that if M and N are finitely generated S -modules such that $M^{(t)} \cong N^{(t)}$, then $M \cong N$. Here $M^{(t)}$ denotes the direct sum of t copies of M .

In Section 3, a generalization of a theorem of Roth [16] is considered. Suppose α_{ij} is an $n_i \times m_j$ matrix over R , $1 \leq i \leq j \leq k$. It is shown that for certain classes of rings (including rings module finite over a commutative

ring), if

$$A = \begin{pmatrix} \alpha_{11} & & 0 \\ & \ddots & \\ 0 & & \alpha_{kk} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha_{11} & & \alpha_{ij} \\ & \ddots & \\ 0 & & \alpha_{kk} \end{pmatrix} \quad (1.1)$$

are equivalent (or satisfy $\text{coker } A \cong \text{coker } B$), then there exist $n_i \times n_i$ matrices γ_{ij} and $m_i \times m_i$ matrices δ_{ij} such that

$$\begin{pmatrix} I & & \gamma_{ij} \\ & \ddots & \\ 0 & & I \end{pmatrix} A = B \begin{pmatrix} I & & \delta_{ij} \\ & \ddots & \\ 0 & & I \end{pmatrix}. \quad (1.2)$$

Note that (1.2) is a linear condition. Roth [16] first proved this for $k=2$ and $R = F[x]$. Also, for $k=2$, Gustafson [9] proved this for an arbitrary commutative ring. Feinberg [6] obtained the result for all k and R a field. See also [8], [10], and [11] for related results. There is a corresponding result for similarity (see Section 4).

A related result holds for decomposition of modules. Suppose S is a ring module finite over a commutative ring and M is a finitely presented S -module. If

$$0 = M_0 \subset \cdots \subset M_k = M \quad \text{and} \quad M \cong \bigoplus_{i=1}^k M_i / M_{i-1}, \quad (1.3)$$

then in fact each M_i is a summand of M . This is proved for $k=2$ and S noetherian in [14] and without the noetherian restriction in [8]. Surprisingly, it appears that one cannot reduce to the case $k=2$.

All rings have 1, and unless otherwise stated modules are right modules.

2. MATRIX EQUIVALENCE

Recall that a ring is semiperfect if finitely generated modules have projective covers (see [2]). Let $J(R)$ denote the Jacobson radical of R . Then R is semiperfect if $R/J(R)$ is semisimple and idempotents modulo $J(R)$ can be lifted [2, Theorem 2.1]. In particular, artinian rings and local rings are semiperfect. Also if R is a commutative noetherian local ring with maximal ideal P which is complete with respect to the P -adic topology and S is a

module finite R -algebra, then S is semiperfect. Recall an R -algebra is an overring S of R with R central in S .

Levy and Robson [13, Theorem 4.3] prove that if $A, B \in R_{n \times m}$, R is semiperfect, and $\text{coker } A \cong \text{coker } B$, then A and B are equivalent. We extend this result to rings S such that 1 is in the stable range of S . (1 is in the stable range of R if $aR + bR = R$ implies $a + bx$ is a unit in R for some $x \in R$.) Rings satisfying this property include:

- (a) Artinian rings (Bass [16, Lemma 11.8]), and so more generally rings R with $R/J(R)$ artinian (including semiperfect rings).
- (b) Module finite algebras over a commutative ring R with $R/J(R)$ von Neumann regular [7] (including R semilocal).
- (c) Module finite algebras over a commutative local-global ring R [4].

R is a local global ring if whenever $f(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ represents a unit locally, then it does globally. Such rings include commutative rings with $R/J(R)$ von Neumann regular, the ring of all algebraic integers, and $S^{-1}R[x]$ with R any commutative ring and S the set of primitive polynomials in $R[x]$. See [4] for a discussion of such rings.

THEOREM 2.1. *Suppose 1 is in the stable range of R_n and R_m , and $A, B \in R_{n \times m}$. If $\text{coker } A \cong \text{coker } B$, then A and B are equivalent.*

Proof. Say the isomorphism from R^n/AR^m to R^n/BR^m is induced by left multiplication by $X \in R_n$. Hence $R^n = XR^n + BR^m$, and so $R_n = XR_n + BR_{m \times n}R_n$. Since 1 is in the stable range of R_n , $X_1 = X + BT$ is a unit for some $T \in R_{m \times n}$. Since X_1 induces the same isomorphism as X , $X_1AR^m = BR^m$. By replacing A with X_1A , we can assume that $AR^m = BR^m$. Thus $A = BC$ and $B = AD$ for some $C, D \in R_m$. Hence $A(I - DC) = 0$, and so $DR_m + L = R_m$, where $L = \{Y \in R_m \mid AY = 0\}$. The stable range condition implies that $D + E$ is a unit for some $E \in L$ and so $B = AD = A(D + E)$ is equivalent to A . ■

B. McDonald has informed me that Warfield [17, Theorem 4] has obtained Theorem 2.1 under the hypothesis that 1 is in the stable range of R and that 1 in the stable range of R implies 1 is in the stable range of R_k [18].

In particular, Theorem 2.1 includes the Levy-Robson result. For algebras over local commutative rings, another proof can be given. This leads to some other interesting consequences. Some preliminary results are needed.

LEMMA 2.2. *Let R be a local noetherian commutative ring with maximal ideal P . Suppose S is a module finite R -algebra and $A, B \in S_{n \times m}$. The*

following are equivalent:

- (a) A and B are equivalent.
- (b) A and B are equivalent over $\hat{S} = S \otimes_R \hat{R}$, where \hat{R} is the completion of R with respect to the P -adic topology.
- (c) A and B are equivalent over $S/P^e S$ for e sufficiently large.

Proof. Obviously (a) implies (b) implies (c). Define $\Delta: S_n \times S_m \rightarrow S_{n \times m}$ by

$$\Delta(X, Y) = XA - BY.$$

By the Artin-Rees lemma (cf. [3, p. 197]), there exists $f = f(A, B)$ such that

$$\text{im } \Delta \cap P^{i+f} S_{n \times m} \subset P^i \text{im } \Delta \quad \text{for } i > 0. \quad (2.1)$$

Now if A and B are equivalent over $S/P^{f+1}S$, there exist $X \in \text{Gl}_n(S)$ and $Y \in \text{Gl}_m(S)$ with $\Delta(X, Y) \equiv 0 \pmod{P^{f+1}S_{n \times m}}$. Thus $\Delta(X, Y) = \Delta(X', Y')$, where $X' \in PS_n$ and $Y' \in PS_m$. Hence $(X - X')A = B(Y - Y')$, and since $X \equiv X - X' \pmod{P}$ and $Y \equiv Y - Y' \pmod{P}$, $X - X'$ and $Y - Y'$ are invertible. ■

The next result is standard. See [10, Lemma 2]. It will also be used in section 3.

LEMMA 2.3. *Let S be a module finite R -algebra. If Ω is a finite subset of S , there exists a noetherian subring R_0 of R such that $R_0[\Omega]$ is a module finite R_0 -algebra. Furthermore, if R is local, R_0 can be chosen local.*

The equivalence result for algebras over local rings follows from Lemmas 2.2 and 2.3. For one can reduce to the noetherian case by Lemma 2.3 and then to the artinian case by Lemma 2.2. An interesting consequence of Lemma 2.2 is:

THEOREM 2.4. *Let R be a local noetherian ring with maximal ideal P . If S is a module finite R -algebra, and M and N are finitely generated S -modules such that either*

- (a) $M/P^e M \cong N/P^e N$ for e sufficiently large, or
- (b) $\hat{M} = M \otimes_R \hat{R} \cong \hat{N}$ as \hat{S} -modules,

then $M \cong N$.

Proof. Choose $A, B \in S_{n \times m}$ such that $M \cong \text{coker } A$ and $N \cong \text{coker } B$. If $e > f(A, B)$ (see 2.1) and $M/P^e M \cong N/P^e N$, then A and B are equivalent over $S/P^e S$, and hence by Lemma 2.2, they are equivalent over S . Thus $M \cong N$. Similarly (b) holds. ■

Theorem 2.4(b) is known. It follows from the facts that \hat{R} is a faithfully flat R -module, $P\hat{R}$ is the maximal ideal of \hat{R} , $\hat{R} = R + P\hat{R}$, and $\text{Hom}_S(M, N) \otimes_R \hat{R} = \text{Hom}_{\hat{S}}(\hat{M}, \hat{N})$. See [1, Lemma 2.4] for a proof of the last property.

Although local rings do not necessarily satisfy the Krull-Schmidt theorem, many of its consequences do hold. For example, if S is a module finite R -algebra, R a local-global ring, then S satisfies the cancellation property for finitely generated modules. ([4, Theorem 2.5]) So if $A \oplus M \cong B \oplus M$ for A, B, M S -modules with M finitely generated, then $A \cong B$. Theorem 2.4 affords a new proof of a result of Goodearl and Warfield [7, Theorem 19] for algebras over local rings.

THEOREM 2.5. *Let S be a module finite R -algebra, R a local ring. If M and N are finitely generated S -modules such that $M^{(t)} \cong N^{(t)}$, then $M \cong N$.*

Proof. As in [7], if $E = \text{End}_S(M)$, it suffices to prove that $D^{(t)} \cong E^{(t)}$ implies $D \cong E$ for D an E -module. Let A and $B \in E_{n \times m}$ with $E \cong \text{coker } A$ and $D \cong \text{coker } B$. Set $A' = \text{diag}(A, \dots, A)$ and $B' = \text{diag}(B, \dots, B)$. If F is a free S -module mapping onto M , then E is a homomorphic image of R -subalgebra of $S_n = \text{End}_S(F)$, and thus by Theorem 2.1, A' and B' are equivalent over E . By passing to a subring as in Lemma 2.3, we can assume A' and B' are equivalent in a subring E_0 which is a module finite algebra over a local noetherian ring R_0 . Say P is the maximal ideal of R_0 .

Let M_1 and N_1 be the modules $\text{coker } A'$ and $\text{coker } B'$ over E_0 . So $M_1^{(t)} \cong N_1^{(t)}$ and $(M_1/P^e M_1)^{(t)} \cong (N_1/P^e N_1)^{(t)}$. Since $E_0/P^e E_0$ is artinian, this implies $M_1/P^e M_1 \cong N_1/P^e N_1$, and so $M_1 \cong N_1$ by Theorem 2.4. By Theorem 2.1, this implies A and B are equivalent over E_0 and hence over E . Thus $D \cong E$ as desired. ■

Estes and the author [4, Theorem 2.11] have extended the above result to local global rings.

3. ROTH'S THEOREM

Let us say that a ring R has the equivalence property if whenever A and B are equivalent matrices as in (1.1), then (1.2) holds. Let W_i and V_i be free R -modules of rank m_i and n_i , respectively, for $1 \leq i \leq k$. Then A and B can be

considered as maps from

$$W = W_1 \oplus \cdots \oplus W_k \quad \text{to} \quad V = V_1 \oplus \cdots \oplus V_k.$$

Set $W'_i = W_1 \oplus \cdots \oplus W_i$ and $V'_i = V_1 \oplus \cdots \oplus V_i$. Let ν be the canonical map from V to V/BW .

LEMMA 3.1. *The following are equivalent:*

- (a) *There exist γ_{ij} and δ_{ij} such that (1.2) holds.*
- (b) *$V'_i \cap BW = BW'_i$ and $\nu(V'_i)$ is a summand of $\nu(V)$.*

Proof. If $k=2$, the result is [8, Lemma 2.2]. For $k>2$, write $W = W'_{k-1} \oplus W_k$ and $V = V'_{k-1} \oplus V_k$. By the result for $k=2$, it suffices to now assume that $\alpha_{1k}, \dots, \alpha_{k-1,k}$ are all 0. The result follows by induction. ■

If for any finitely presented R -module M satisfying (1.3), each M_i is a summand, say R has the extension property.

LEMMA 3.2. *Suppose R is a right artinian ring. Then the extension property implies the equivalence property.*

Proof. Let A and B be as in (1.1). If A and B are equivalent, then

$$\nu(V) \cong \bigoplus_{i=1}^k V_i / \alpha_{ii} W_i.$$

Now

$$\nu(V'_i) / \nu(V'_{i-1}) \cong V_i / V_i \cap (BW + V'_{i-1})$$

Hence by calculating the composition length of $\nu(V)$, $V_i \cap (BW + V'_{i-1}) = \alpha_{ii} W_i$, and hence by a straightforward induction argument, $V'_i \cap BW = BW'_i$. Thus 3.1(b) holds. Hence by Lemma 3.1, the equivalence property holds. ■

Since any semisimple artinian ring has the extension property trivially, the next result follows.

THEOREM 3.3. *A semisimple artinian ring has the equivalence property.*

Hartwig [11] and Gustafson and Zelmanowitz [10] proved this for $k=2$. The converse of Lemma 3.2 holds. In fact more is true.

LEMMA 3.4. *Suppose that 1 is in the stable range of S . Then the equivalence property implies the extension property.*

Proof. Suppose M is finitely presented and $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that

$$M \cong \bigoplus_{i=1}^k M_i / M_{i-1}.$$

Hence M_i / M_{i-1} is finitely presented. Thus there exist finitely generated free modules V_i, W_i , $1 \leq i \leq k$, and $\alpha_{ii} \in \text{Hom}(W_i, V_i)$ such that $M_i / M_{i-1} \cong V_i / \alpha_{ii} W_i$. Choose $\varphi_i \in \text{Hom}(V_i, M_i)$ that induce these isomorphisms. Thus $\varphi_i \alpha_{ij} W_j \subset M_{j-1} = \varphi_1(V_1) + \cdots + \varphi_{j-1}(V_{j-1})$. Since each W_j is free, there exist $\alpha_{ij} \in \text{Hom}(W_j, V_i)$, $1 \leq i < j$, such that

$$-\varphi_i \alpha_{ij} = \sum_{i < j} \varphi_i \alpha_{ij}.$$

Set $W = \bigoplus W_i$ and $V = \bigoplus V_i$. Define $B \in \text{Hom}(W, V)$ by (1.1). Let $\sigma \in \text{Hom}(V, M)$ be defined by

$$\sigma(v_1 + \cdots + v_k) = \varphi_1(v_1) + \cdots + \varphi_k(v_k).$$

It is straightforward to verify that σ is surjective, $\ker \sigma = BW$, and so σ induces an isomorphism of $\text{coker } B$ with M . Furthermore M_i and $\nu(V_1 \oplus \cdots \oplus V_i)$ correspond under this isomorphism.

Now $\text{coker } B \cong \text{coker } A$, where $A = \alpha_{11} \oplus \cdots \oplus \alpha_{kk}$. Hence by Theorem 2.1, A and B are equivalent. If S has the equivalence property, then $\nu(V_1 \oplus \cdots \oplus V_i)$ is a summand of $\nu(V)$ by Lemma 3.2. Hence M_i is a summand of M . ■

Suppose now that S is a module finite R -algebra, R an artinian ring.

LEMMA 3.5. *S satisfies the extension property.*

Proof. Let M and M_i be as in (1.3). Thus for N any S -module,

$$\text{Hom}_S(M, N) \cong \bigoplus_{i=1}^k \text{Hom}_S(M_i / M_{i-1}, N). \quad (3.1)$$

For each i, j , there is an exact sequence

$$0 \rightarrow \text{Hom}(M_i/M_{i-1}, M_j) \rightarrow \text{Hom}(M_i, M_j) \xrightarrow{r_{ij}} \text{Hom}(M_{i-1}, M_j), \quad (3.2)$$

where r_{ij} is the restriction mapping. Let a_{ij} and b_{ij} denote the lengths (as R -modules) of $\text{Hom}(M_i/M_{i-1}, M_j)$ and $\text{Hom}(M_i, M_j)$, respectively. By (3.2), $b_{ij} \leq a_{ij} + b_{i-1, j}$, and so

$$\sum_{i=1}^k b_{ij} \leq \sum_{i=1}^k a_{ij} + \sum_{i=1}^{k-1} b_{ij}.$$

Hence $b_{kj} \leq a_{ij} + \cdots + a_{kj}$. By (3.1), this must be actual equality. Thus equality holds in all the equations. This implies r_{ij} is surjective, and yields M_{i-1} is a summand of M_i for each i . This is the desired result. ■

If S is any artinian ring, the above result is still true for bimodules. In that case, $\text{Hom}(M, N)$ is an S -module, and one can argue as above. It is still open whether Lemma 3.5 holds for artinian rings. We can now prove the main result of this section.

THEOREM 3.6. *Suppose S is a module finite R -algebra, R commutative. Then S has the extension and equivalence properties.*

Proof. For the equivalence property, one can pass to a finitely generated subring. Thus we can assume R is noetherian by Lemma 2.3. Now (1.3) can be expressed as a linear system of equations. In particular (1.3) holds if and only if there exist γ_{ij} and δ_{ij} such that

$$\alpha_{ij} = \gamma_{ij} \alpha_{jj} - \sum_{l=i}^{j-1} \alpha_{il} \delta_{lj} \quad (3.3)$$

for $1 \leq i < j \leq k$. The right hand side is an R -linear expression, and so (3.3) has a solution if and only if it does over S_P for each maximal ideal P [3, p. 88]. By [8, Lemma 3.3(ii)], (3.3) has a solution over S_P if and only if it does over $S_P/P^e S_P$ for each e . However, $S_P/P^e S_P$ is artinian, and so by Lemma 3.5 has the extension property. Thus by Lemma 3.2, it has the equivalence property. So (3.3) is solvable, and S has the equivalence property.

Since M_i/M_{i-1} is finitely presented, M_{i-1} is a summand of M_i if and only if this is true locally (cf. [3, p. 90]). Thus for the extension property, we can

assume R is local. By Lemma 3.4, the equivalence property now implies the extension property. ■

Some noncommutative results can be obtained. Hartwig [11], however, showed that if $ab = 1 \neq ba$ for some $a, b \in R$, then R does not have the equivalence property. Call R directly finite if $ab = 1$ implies $ba = 1$. In [8], it is shown that if R_n is a directly finite von Neumann regular ring for all n , then R has the equivalence property for $k=2$. The proof can be modified for arbitrary k .

Suppose that R is an artinian principal ideal ring. Then R is an injective R -module. For such rings, we have:

THEOREM 3.7. *R has the extension and equivalence properties.*

Proof. By Lemma 3.2, it suffices to prove the extension property. So let the notation be as in (1.3). By refining the series, we can assume M_1 is indecomposable. If M_1 is faithful, then by the classification of R -modules (see [12, p. 79]), M_1 is projective and hence injective by the remarks preceding the theorem. Hence M_1 is a summand of M .

So let I be the annihilator of M_1 . We certainly can assume that M is faithful and so $MI \neq 0$. Let \bar{M}_i denote the image of M_i under the mapping M onto M/MI . By (1.3), if l denotes the composition length,

$$l(\bar{M}) = \sum_{i=1}^k l(M_i/M_iI + M_{i-1}).$$

However, $M_i/(M_iI + M_{i-1}) \cong (M_i/M_iI)/(M_iI + M_{i-1}/M_iI)$, and so

$$l(\bar{M}) = \sum_{i=1}^k l(M_i/M_iI) - \sum_{i=1}^k l(M_{i-1}/M_{i-1} \cap M_iI).$$

Since $M_k = M$, this yields

$$\sum_{i=1}^{k-1} l(M_i/M_iI) = \sum_{i=1}^{k-1} l(M_i/M_i \cap M_{i+1}I).$$

As each term on the right hand side is a homomorphic image of the corresponding term on the left, this implies $M_iI = M_{i+1}I \cap M_i$ for each i .

This implies that $M_i I = M_i \cap MI$, and hence

$$\overline{M} \approx \bigoplus_{i=1}^k \overline{M}_i / \overline{M}_{i-1}.$$

Since $l(\overline{M}) < l(M)$, by induction each \overline{M}_i is a summand of \overline{M} . In particular, there exists a submodule L of M with $MI \subset L$ so that $\overline{M} = \overline{M}_1 \oplus \overline{L}$. Hence $M = M_1 + L$ and $M_1 \cap L \subset M_1 \cap MI = M_1 I = 0$. Thus M_1 is a summand of M whether or not M_1 is faithful.

Now passing to M/M_1 , it follows by induction that each M_i/M_1 is a summand and so M_i is a summand. ■

4. SIMILARITY PROPERTY

Suppose A and B are as in (1.1) except that $m_i = n_i$. Thus α_{ij} , A , and B are all square matrices. We say R has the similarity property if whenever A and B are similar and are as above, then there exist $n_i \times n_i$ matrices γ_{ij} such that

$$\begin{pmatrix} I & \gamma_{ij} \\ 0 & I \end{pmatrix} A = B \begin{pmatrix} I & \gamma_{ij} \\ 0 & I \end{pmatrix}. \quad (4.1)$$

This can be shown to be equivalent to solving the system of equations

$$\alpha_{ij} = \delta_{ij} \alpha_{jj} - \alpha_{ii} \delta_{ij} \quad \text{for } 1 \leq i < j \leq k, \quad (4.2)$$

where δ_{ij} is an $n_i \times n_j$ matrix over R . Feinberg [5] shows this for R a field, but the proof does not depend on this fact. The next result shows how the similarity property relates to the other properties.

THEOREM 4.1. *Let $R[x]$ denote the ring of polynomials over R , where x commutes with R . If $R[x]$ has the extension or equivalence property, then R has the similarity property.*

Proof. Suppose $R[x]$ has the extension property. Let A and B be as above, and set $n = n_1 + \cdots + n_k$. These determine two $R[x]$ -module structures M and M' on R^n by declaring $vx = Bv$ or Av , respectively, for $v \in R^n$. Let $0 = V_0 \subset V_1 \subset \cdots \subset V_k = R^n$ be the corresponding invariant subspaces. If

A and B are similar, then

$$M \cong \bigoplus_{i=1}^k V_i / V_{i-1} \quad (\text{as } R[x]\text{-modules}).$$

By the extension property, this implies each V_i is an $R[x]$ -summand of M . Hence there exists an $R[x]$ isomorphism σ from M' to M mapping V_i to V_i and which is the identity on V_i / V_{i-1} . Thus $\sigma A = B\sigma$ and σ is of the form in (4.1).

Now suppose that $R[x]$ has the equivalence property. If A and B are as above and similar, then $A - xI$ and $B - xI$ are equivalent. The equivalence property now implies there exist $U, V \in \text{Gl}_n(R[x])$ (where $n = n_1 + \cdots + n_k$) such that $U(A - xI) = (B - xI)V$ and U and V are as in (1.2). Write $U = U_0 + xU_1 + \cdots + x^l U_l$ and $V = V_0 + \cdots + x^l V_l$, where $U_i, V_i \in R_n$. Equating coefficients, we obtain

$$\begin{aligned} U_0 A &= B V_0, \\ U_1 A - U_0 &= B V_1 - V_0, \\ &\vdots \\ U_l A - U_{l-1} &= B V_l - V_{l-1}, \\ U_l &= V_l. \end{aligned}$$

Multiplying the equations on the left by I, B, \dots, B^l and adding yields

$$(U_0 + B U_1 + \cdots + B^l U_l) A = B (U_0 + B U_1 + \cdots + B^l U_l).$$

Since U is as in (1.2), $W = U_0 + B U_1 + \cdots + B^l U_l$ has the form of (4.1), and the similarity property holds. ■

COROLLARY 4.2. *If R is commutative and S is a module finite R -algebra, then S has the similarity property.*

Proof. $S[x]$ is a module finite $R[x]$ -algebra, and so by Theorem 3.6, $S[x]$ has the equivalence property. Now apply Theorem 4.1. ■

It is unknown whether division rings have the similarity property. If A is algebraic over a division ring D , let $f(x) \in D[x]$ be a central polynomial with $f(A) = 0$. Then A and B determine two $D[x]$ -module structures on D^n

as in the proof of Theorem 4.1. If A and B are similar, these are in fact $D[x]/(f(x))$ -modules. Since $D[x]/(f(x))$ is an artinian principal ideal ring, Theorem 3.7 and the proof of Theorem 4.1 show that A and B have the similarity property.

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